

# CALCULATIONS OF ELASTIC WAVE SCATTERING FROM VOIDS AND CRACK-LIKE DEFECTS BY THE METHOD OF OPTIMAL TRUNCATION

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## ABSTRACT

Visscher's method of optimal truncation for solving elastic wave scattering problems has been applied to voids having surface cracks and sharp edges including some for which experimental data are becoming available. A number of interesting aspects of these results will be discussed and comparisons with existing experimental data presented.

## INTRODUCTION

One of the key elements in the development of quantitative ultrasonic NDE is the elastic wave scattering problem. It is essential that practical and reliable methods be found that accurately predict the scattering of elastic waves from flaws of arbitrary size and shape. One method offering some promise in that direction is the method of optimal truncation<sup>1</sup> (MOOT) a least squares boundary residual method originally proposed for electromagnetic scattering<sup>2</sup>.

Although the method's utility has been demonstrated for a number of non-spherical scatterers<sup>1,3</sup>, there are cases when its convergence to the exact solution becomes difficult, if not impossible, to achieve. In this talk I will first briefly introduce the method and apply it to an example for which experimental data are becoming available. I will then describe a modification that appears to overcome some of the convergence difficulties and illustrate with a specific example. The modified method will then be applied to two examples of practical interest.

## THEORY

The basic scattering problem we consider is a void imbedded in a homogeneous isotropic host with material properties  $\lambda$ ,  $\mu$ , and  $\rho$ , where  $\lambda$  and  $\mu$  are the Lamé parameters relating stress to strain and  $\rho$  is the density. In terms of these parameters, we seek to solve the elastic wave equation (time dependence,  $e^{-i\omega t}$ , assumed)

$$(\lambda + 2\mu)\nabla\nabla\cdot\mathbf{u} - \mu\nabla\times(\nabla\times\mathbf{u}) + \rho\omega^2\mathbf{u} = 0, \quad (1)$$

for the displacement field,  $\mathbf{u}$ .

To implement MOOT we first expand the displacement field in some complete set of functions. A suitable set of basis functions for 3-dimensional scattering problems are the solutions to Eq. (1) in spherical coordinates,

$$u_{1\ell m} = \frac{1}{k} \nabla\{Z_\ell(kr)Y_{\ell m}(\theta, \phi)\}, \quad (2a)$$

$$u_{2\ell m} = i \frac{1}{\sqrt{\ell(\ell+1)}} r \times \nabla\{Z_\ell(kr)Y_{\ell m}(\theta, \phi)\}, \quad (2b)$$

$$u_{3\ell m} = \frac{1}{k} \nabla \times u_{2\ell m} \quad (2c)$$

where  $Z_\ell$  are the spherical Hankel (outgoing waves) or Bessel (standing waves) functions and  $Y_{\ell m}$  are

the spherical harmonics. Also in Eq. (2),  $k$  is the longitudinal wave vector,  $k = \omega\sqrt{\frac{\rho}{\lambda+2\mu}}$ , and  $K$  is the transverse wave vector,  $K = \omega\sqrt{\frac{\rho}{\mu}}$ . The complete solution in terms of these basis functions is given by,

$$\mathbf{u} = \mathbf{u}_0 + \sum_{\ell m} a_{\ell m} u_{\ell m} \quad (3)$$

where  $\mathbf{u}_0$  denotes the incident wave, presumably known for the scattering problem and the summation contains the unknown coefficient  $a_{\ell m}$  which completely describes the scattered wave. Of the several possibilities which exist for determining the  $a_{\ell m}$ , at least in an approximate sense, MOOT selects the most obvious for a demonstrably convergent procedure. When the infinite series in Eq. (3) is truncated, as must be done in practice, an error is encountered in satisfying the boundary conditions. MOOT optimizes the approximate solution by a least squares minimization of that error. The result of that minimization for a void scatterer, is the set of linear equations,

$$p'_{\ell m, p'} G_{p\ell m, p'} a_{p'\ell m} + d_{p\ell m} = 0 \quad (4)$$

where

$$G_{p\ell m, p'\ell m} = \int_{\text{void}} dS \mathbf{t}_{p\ell m}^* \cdot \mathbf{t}_{p'\ell m}, \quad (5)$$

$$d_{p\ell m} = \int_{\text{void}} dS \mathbf{t}_{p\ell m}^* \cdot \mathbf{t}_0, \quad (6)$$

and  $\mathbf{t}_{p\ell m}$  and  $\mathbf{t}_0$  are the tractions on the surface of the void derived, respectively, from the displacements  $u_{p\ell m}$  and  $u_0$ . In this particular case, since the scattered wave is outgoing far from the scatterer, the Hankel functions must be used in Eq. (2).

When considering the convergence of the method, we examine the following quantities: (1) the normalized boundary residual,

$$I/I_0 = \int_{\text{void}} dS \left| \sum_{\ell m} a_{\ell m} \mathbf{t}_{p\ell m} + \mathbf{t}_0 \right|^2 / \int_{\text{void}} dS |\mathbf{t}_0|^2 \quad (7)$$

and

(2) the total normalized cross-section,

$$P = \sum_{\ell m} \{|a_{1\ell m}|^2 + \frac{k}{K}(|a_{2\ell m}|^2 + |a_{3\ell m}|^2)\} / \pi(ka)^2 \quad (8)$$

and (3) the optical theorem

$$P = \text{Im}(F) \quad , \quad (9)$$

where  $F$  is the amplitude of the forward scattered wave,

$$F = 4\pi \sum_{\ell m} (-i)^{\ell+1} a_{\ell m} Y_{\ell m}(\theta_0, \phi_0) \quad . \quad (10)$$

#### DISCUSSION

For our first example we consider the scatterer shown in Fig. 1. This defect, consisting of a spherical void of radius  $a$  with a hemispherical bubble of radius  $\frac{1}{2}a$  on its surface, is significant since it has been studied earlier both theoretically<sup>4</sup> and experimentally<sup>5</sup>. The previous theoretical calculations<sup>4</sup> based on the DWBA agree very well with the results obtained using MOOT shown in Figs. 2-4 for frequencies corresponding to

$ka \lesssim 3$ . Such agreement was not unexpected and tends to confirm both methods. Most significantly, there is quite good agreement between both theories and the experimental results<sup>5</sup>. It is also interesting to note that the rather complicated appearing structure of the results shown in Figs. 2-4 can be interpreted in terms of the multiple interferences resulting from the presence of the bubble. For example, the reflection and transmission of the creep wave at the corner where the bubble intersects the sphere explains essentially all of the structure in Fig. 2.

As I mentioned earlier, there are cases when convergent solutions are not obtained with MOOT. These generally occur for large aspect ratios; that is, whenever the largest and smallest dimensions of the scatterer differ by more than about a factor of 2. For such scatterers, the large variation in the Hankel functions (used to represent the scattered field) over the surface of the scatterer makes it extremely difficult to satisfy the boundary conditions. In fact at large enough aspect ratios, the system of equations, (Eq. (4)), becomes so in-conditioned that not even a reasonably good approximation to the exact solution is possible. One way to circumvent this difficulty is to use functions that are well-behaved along the scatterer's boundary to represent the near-field and match those up to the outgoing waves at some other fictitious boundary where the Hankel functions are well-behaved. In Fig. (5), I show how this can be done for the case of an oblate spheroidal void. The spheroid is enclosed by a sphere of radius equal to the major axis of the spheroid. For the near field, between the sphere and the spheroid, the regular Bessel functions are used to expand the displacement and traction fields, with coefficients  $b_{\ell m}$  in the upper half and  $c_{\ell m}$  in the lower half. Outside the sphere, the Hankel functions are used with coefficients  $a_{\ell m}$  as before. One now seeks to satisfy the boundary conditions: (1) zero traction along the surface of the spheroid and (2) continuity of traction and displacement along the spherical boundary. Although this approach has increased the number of unknowns ( $a_{\ell m}$ ,  $b_{\ell m}$ ,  $c_{\ell m}$ ) by a factor of 3, the matrices are now well-conditioned and the convergence properties improved considerably. Before continuing, I want to point out that Bill

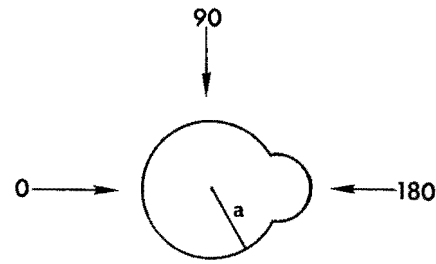
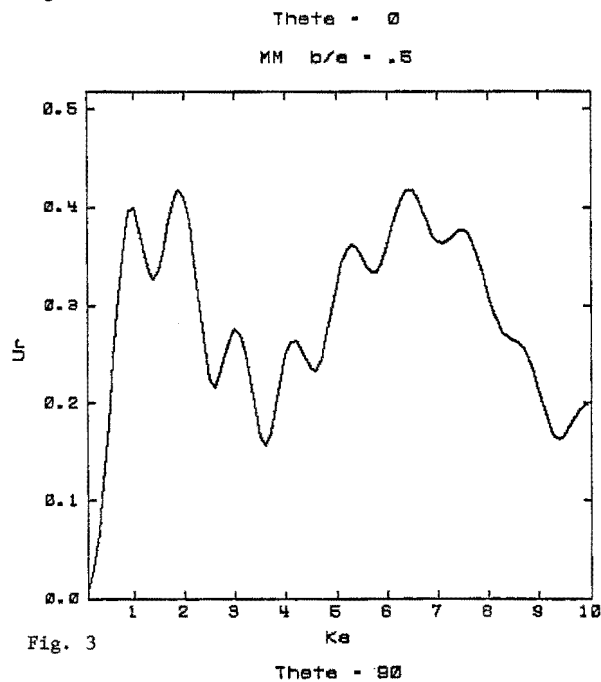
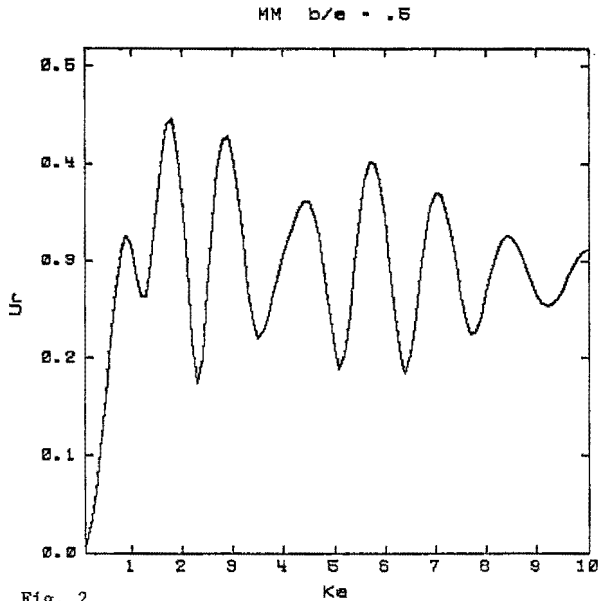


Fig. 1 Spherical void with hemispherical bubble.



Figs. 2-3 Backscattered amplitude from void in Fig. 1 for waves incident at 0° and 90°.

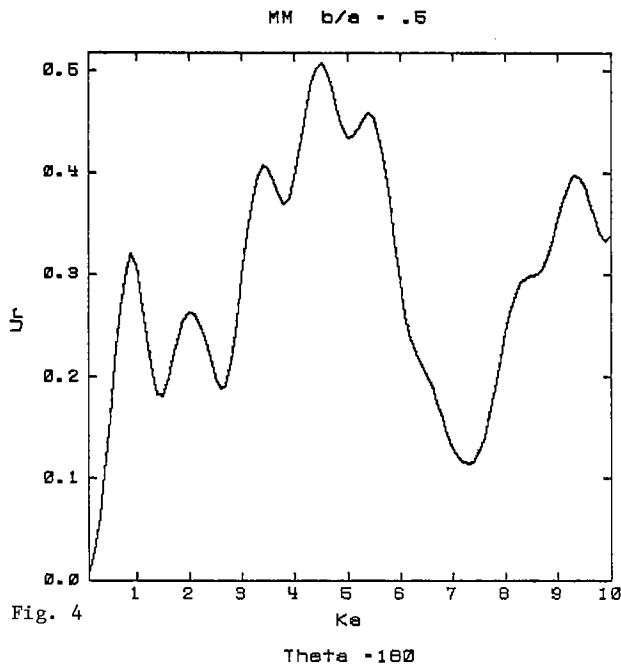


Fig. 4 Backscattered amplitude from void in Fig. 1 for waves incident at  $180^\circ$ .

Visser<sup>6</sup> has applied essentially this same approach to circular cracks. For that problem he reasoned that one requires different expansions above and below the crack in order that there be a crack opening displacement, and further that these expansions must be well-behaved at the origin. By collapsing the spheroid in Fig. 5, one obtains the circular crack as modeled by Visser. To show that this modified version of MOOT has some validity, I show in Figs. (6-8) comparisons with results based on the original version, Eq. (4), for a spheroid with  $b/a = 0.7$ .

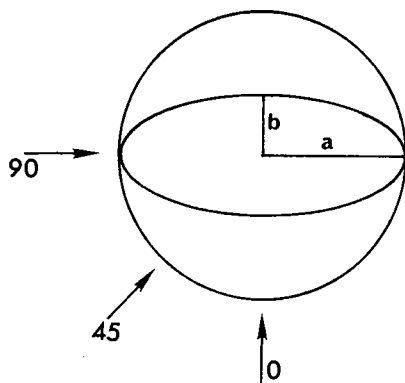
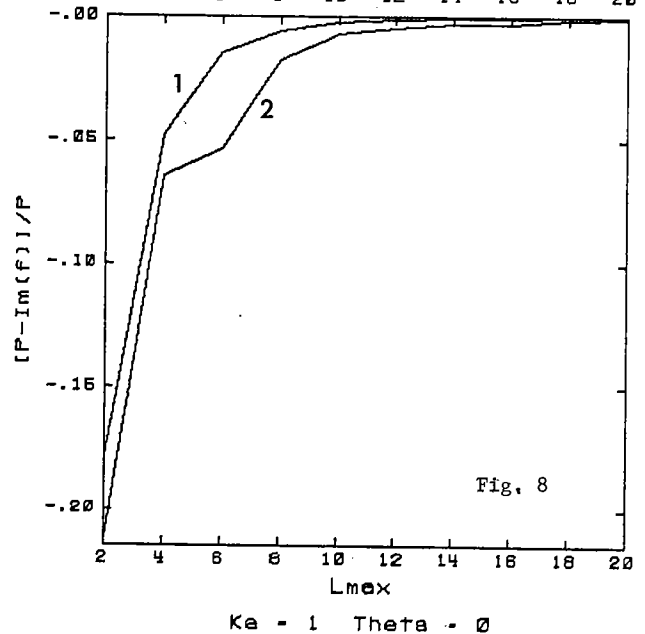
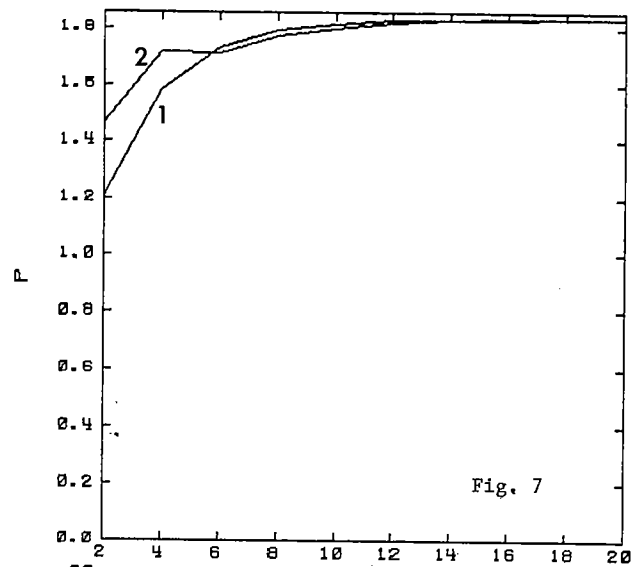
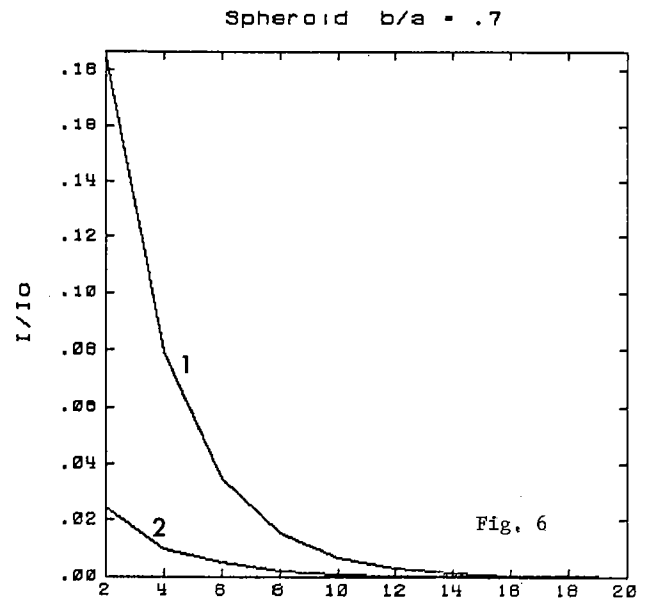


Fig. 5 Spheroidal void enclosed by sphere.

Fig. 6-8 Convergence plots comparing the original (1) and modified (2) versions of MOOT as discussed in the text.  $L_{max}$  is the highest order Bessel function used in the expansions.



Figs. 6-8 show that the two versions of MOOT give the same results and that for that aspect ratio converge at about the same rate. The modified version, however, has a clear advantage over the original as the spheroid becomes more oblate. This is demonstrated by the comparisons in Fig. 9. Not included are corresponding comparisons for the total cross section and the optical theorem, but the results are essentially the same.

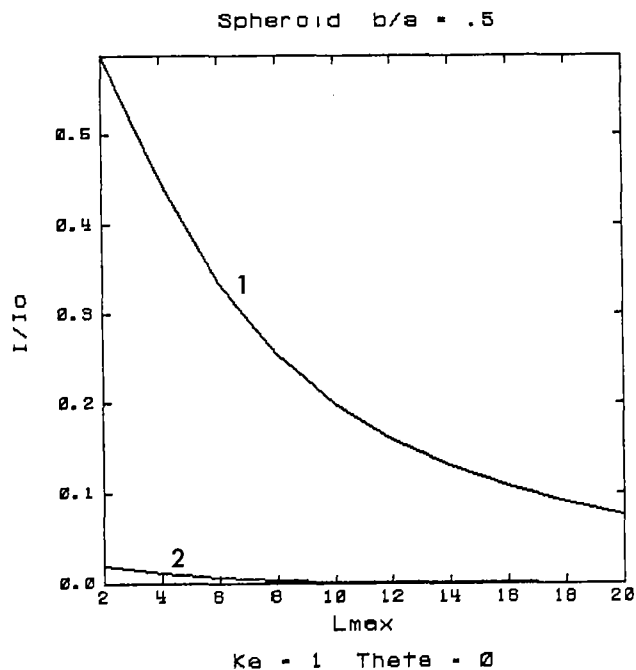


Fig. 9 Plots comparing the convergence of the boundary residual obtained using the original (1) and modified (2) versions of MOOT. Lmax is the highest order Bessel function used in the expansions.

Some results for the 2:1 oblate spheroid are shown in Figs. 10-12.

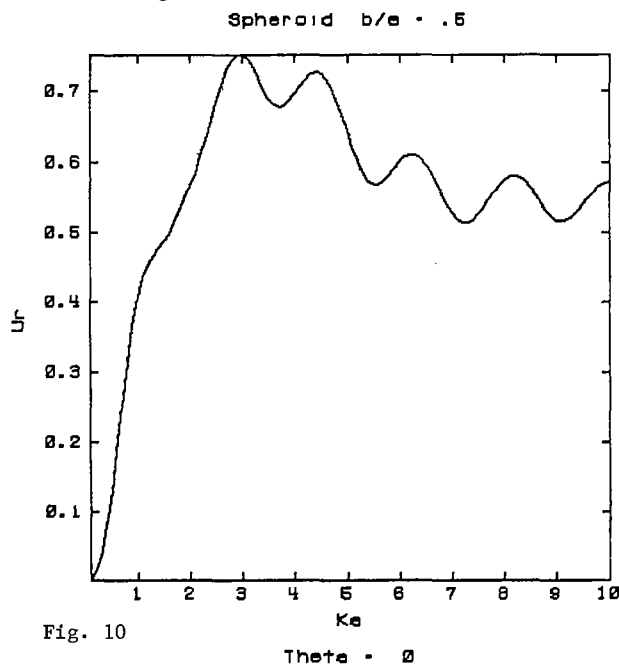


Fig. 10

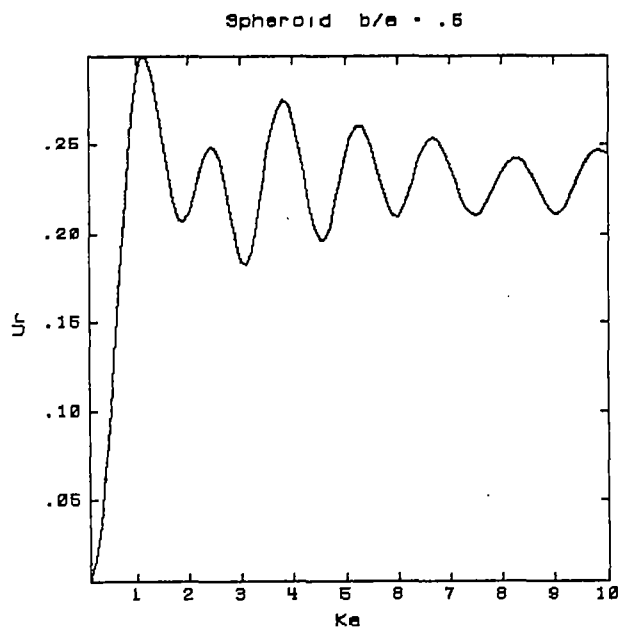


Fig. 11

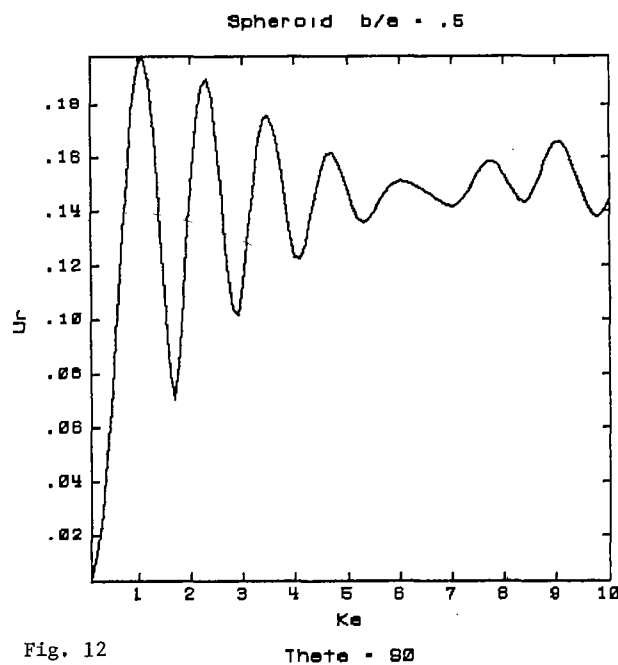


Fig. 12

Figs. 10-12 Backscattered amplitudes from a 2:1 spheroidal void obtained using the modified version of MOOT. Theta is the angle of incidence as indicated in Fig. 5.

By collapsing the spheroid in Fig. 5, one obtains a circular crack. Results for the crack are shown in Figs. 13-15. The results in Fig. 13 for normal incidence are in excellent agreement with the exact results of Mal<sup>7</sup>. For frequencies below  $ka = 2$ , the results in Figs. 13-15 are in agreement with Visscher's<sup>6</sup>.

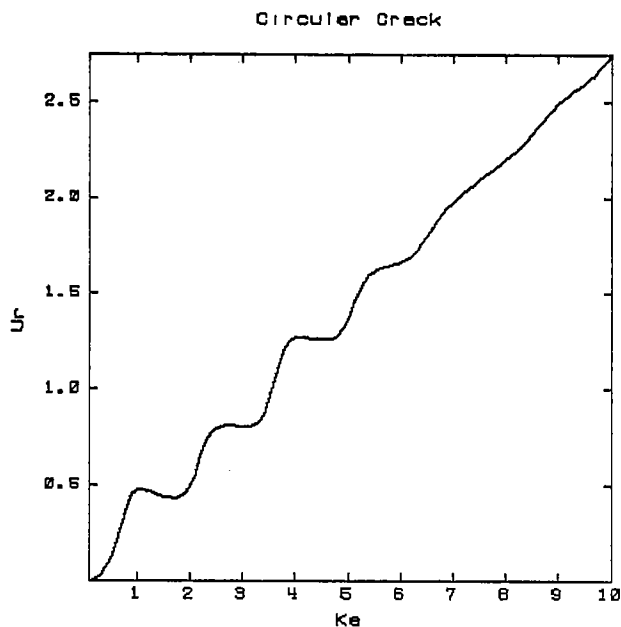


Fig. 13 Theta = 0

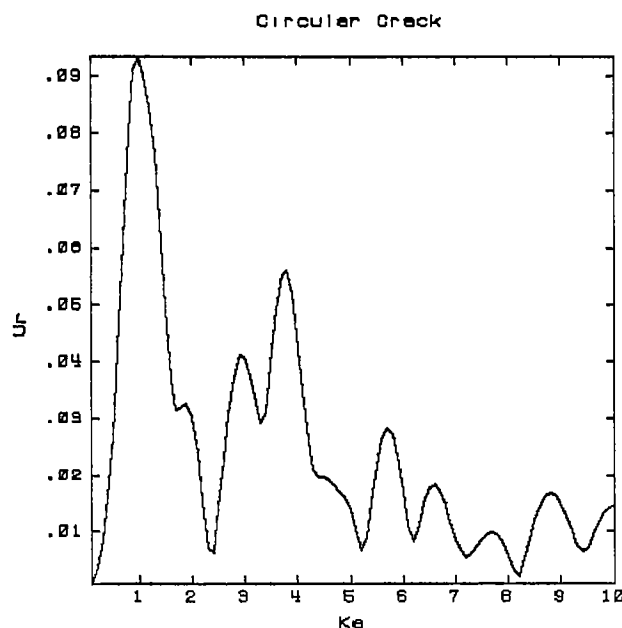


Fig. 15 Theta = 90

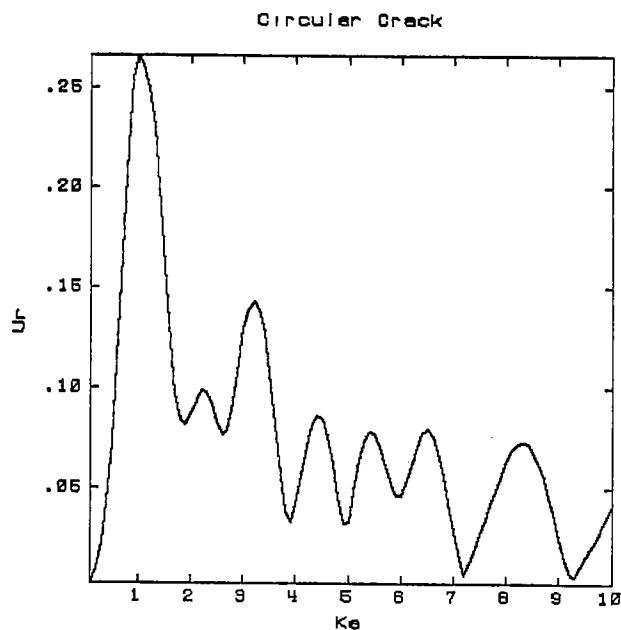


Fig. 14 Theta = 45

Figs. 13-15 Backscattered amplitudes from a circular crack obtained using the modified version of MOOT. Theta is the angle of incidence as indicated in Fig. 5.

In conclusion I would like to say that MOOT appears to have a much wider range of applicability than one might have expected, both in terms of the broad frequency range and in the complexity of the scatterers. A final example which really tests the method is the Saturn ring defect shown in Fig. 16. Some initial comparisons with Bernie Tittmann's<sup>8</sup> experimental results are favorable suggesting a certain degree of correctness in the results shown in Fig. 17.

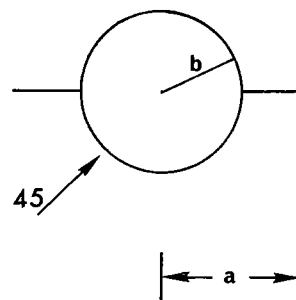


Fig. 16

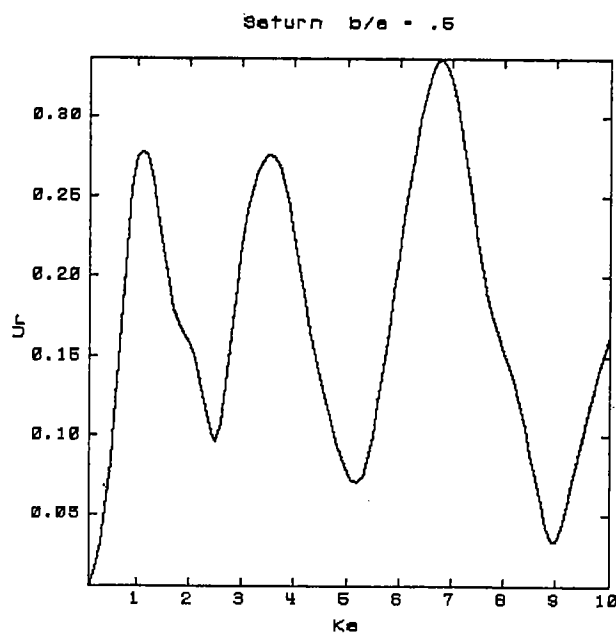


Fig. 17 Theta = 45

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## SUMMARY DISCUSSION

Bruce Thompson, Chairman (Rockwell Science Center [now Ames Laboratory]): We do have a few minutes. Therefore, I suggest we open up questions to any of the first three papers.

Laszlo Adler (Ohio State University): Jon, on the crack nomenclature; the penny-shape crack, I was wondering is that a closed crack or does that have some finite opening?

Jon Opsal (Lawrence Livermore Laboratory): It starts out as a closed crack, and then it opens under load. It's a mathematical crack. I have made real cracks. I have said these things have a little opening to begin with, and you can't tell the differences. I have taken an oblique spheroid that has a very large aspect ratio, and it approaches these results with the crack.

Laszlo Adler: I mean we make these little things that are kind of like pill boxes. They are not really cracks.

Jon Opsal: If you make the aspect ratio large enough, you have a crack.

Laszlo Adler: There is no real opening there?

Jon Opsal: No. A zero volume defect.

Roger Chang (Science Center): On a rectangle, what is plotted in on the last viewgraph?

Vasundara Varadan (Ohio State University): The surface stress.

Roger Chang: Would you show it again, please? The abscissa is the distance along the -

Vasundara Varadan: Along the contour.

Roger Chang: The defect, is it a cube?

Vasundara Varadan: It's a square cylinder. This is a two-dimensional problem. And the wave length was perpendicular to the cylinder. What I have plotted is the surface strength. So CE/CT is equal to one. The maximum is at 0.125. The maxima is at the four corners.

William Pardee (Science Center): I have a question for Vasundara. I had trouble keeping track of various levels of approximation you described. Did you report numerical results more general than the translated T-matrix?

Vasundara Varadan: The quantities that I charted as Sigma were exact results using the full expression for the T-matrix, taking into account all the scattering. I inverted the whole matrix. What I called Sigma hat was allowing the singly spheres. That means I added the amplitudes rather than the cross sections from each one of the spheres or spheroids, so those were the three levels. One was what I called -- which is a full T-matrix. The second one was where you added just two amplitudes to the two singly scattered waves and don't let interfere all but add the two cross sections. It seems like Sigma hat is pretty close to the real Sigma. It's good enough to let them interfere.

Unidentified Speaker: It might be rather useful if these results were taken and put in a time domain with a simple Fourier transform; it offers a little different kind of intuition.

Jon Opsal: I gave my result to Dick Elsley. He has transformed them into the time domain.

Unidentified Speaker: I have a question on the first paper. Were those calculations also for elastic spheres immersed in water?

Vasundara Varadan: No. They were for spherical cavities in elastic solid.

Bruce Thompson, Chairman: I have one question to ask either Bill or Jon. It seems to me, that from the material science point of view a very important problem is a crack that grows from the inclusion, breaks the inclusion and breaks out into the host. You didn't have your host being broken. The Saturn Ring. That's as poor -

Jon Opsal: A cavity with a crack around it.

Bruce Thompson, Chairman: And Bill had a solid medium with a crack on the inside. What are the problems when you try to put that all together and have a crack growing from the inside into the host medium. Is that an order of magnitude or more difficult or just is that simple extrapolation?

Jon Opsal: It's more bookkeeping.

Bruce Thompson, Chairman: A lot more bookkeeping or a little more bookkeeping?

Jon Opsal: No, it's not that much more.

Bruce Thompson, Chairman: One final question.

Chris Burger (Iowa State): Bruce, you were talking about from a theoretical standpoint. I have heard people talking at weld meetings a lot about the significance of porosity. What they observe under the microscope is that the pores are not individual hollow spheres, but the residual stresses - or somewhere in the welding process those pores actually become little spheres with halo-like cracks around them. And when Bill Visscher puts his problem on the board - there is somebody addressing that problem - and then he put the whole thing in a bath of water, and it didn't solve the problem. But could you translate that, essentially, to that kind of problem where you have the hollow sphere with a halo-like crack around -

Bruce Thompson, Chairman: I believe that's what Jon calculated. That is the Saturn Ring defect, I believe.

Jon Opsal: That's just exactly what I was showing.

Bruce Thompson, Chairman: I think Bernie Tittmann will show some experimental results on just that kind of sample.